Midsegments of Triangles

Geometry Unit 5: Lesson 1

Midsegment of a triangle – A segment connecting the midpoints of two sides of a triangle.

Triangle Midsegment Theorem
If a segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side and half the length.

<table>
<thead>
<tr>
<th>If . . .</th>
<th>Then . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$ is the midpoint of $CA$ and</td>
<td>$DE \parallel AB$ and</td>
</tr>
<tr>
<td>$E$ is the midpoint of $CB$</td>
<td>$DE = \frac{1}{2}AB$</td>
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1. What are the three pairs of parallel sides in $\triangle DEF$?

2. What is the $m\angle VUO$ in the figure? Explain your reasoning.
3. In \( \triangle QRS \), \( T \), \( U \), and \( B \) are midpoints. What are the lengths of \( TU \), \( UB \), and \( QR \)?

4. In the figure, \( AD = 6 \) and \( DE = 7.5 \). What are the lengths of \( DC \), \( AC \), \( EF \), and \( AB \)?

5. Find the value of \( x \).

6. Points \( E \), \( D \), and \( H \) are the midpoints of the sides of \( \triangle TUV \). \( UV = 80 \), \( TV = 100 \), and \( HD = 80 \). Find the perimeter of \( \triangle HED \).
1. \( \triangle ABC \) has vertices \( A(0, 0), B(4, 4), \) and \( C(8, 2) \). Use \( \triangle ABC \) and coordinate geometry to demonstrate the Triangle Midsegment Theorem.

a) Find the vertices of \( D \), the midpoint of \( \overline{AB} \).
b) Find the vertices of \( E \), the midpoint of \( \overline{BC} \).
c) Show \( \overline{DE} \parallel \overline{AC} \).
d) Show \( DE = \frac{1}{2} AC \).
Reflection:
1. In ∆JLK, three midsegments are drawn.
   a) How many congruent triangles are located in the figure?

   b) How does the perimeter of ∆NOM compare to
   the perimeter of ∆JLK?

   c) How does the area of ∆NOM compare to
   the perimeter of ∆JLK?
Equidistant – A point is equidistant from two objects if it is the same distance from each object.

Find all points equidistant from points $A$ and $B$. (Hint: Use a construction.)

**Perpendicular Bisector Theorem**

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

*If...*

$\overrightarrow{PM} \perp \overline{AB}$ and $MA = MB$

*Then...*

$PA = PB$

**Converse of Perpendicular Bisector Theorem**

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

*If...*

$PA = PB$

*Then...*

$\overrightarrow{PM} \perp \overline{AB}$ and $MA = MB$
Find all points equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$. (Hint: Use a construction.)

Distance from a point to a line – The length of the perpendicular segment from the point to the line.

**Angle Bisector Theorem**

If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

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<tr>
<td>$\overrightarrow{QS}$ bisects $\angle PQR$, $SP \perp \overrightarrow{QP}$, and $SR \perp \overrightarrow{QR}$</td>
<td>$SP = SR$</td>
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**Converse of Angle Bisector Theorem**

If a point in the interior of an angle is equidistant from the sides of the angle, then the point is on the angle bisector.

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<tr>
<td>$SP \perp \overrightarrow{QP}$, $SR \perp \overrightarrow{QR}$, and $SP = SR$</td>
<td>$\overrightarrow{QS}$ bisects $\angle PQR$</td>
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1. What is the length of $\overline{AB}$?

2. What is the length of $\overline{QR}$?

3. What is the length of $\overline{RM}$?

4. What is the length of $\overline{FB}$?
5. a) A park director wants to build a T-shirt stand equidistant from the Rollin’ Coaster and the Spaceship Shoot. What are the possible locations of the stand? Explain.

b) Suppose the director wants the T-shirt stand to be equidistant from the paddle boats and the Spaceship Shoot. What are the possible locations?

c) Can you place the T-shirt stand so that it is equidistant from the paddle boats, the Spaceship Shoot, and the Rollin’ Coaster? Explain.

Reflection:
1. a) If a point is on the perpendicular bisector of a segment, it is __________________ from the endpoints of the segment.
   b) If a point is on the bisector of an angle, it is __________________ from sides of the angle.
Incenter & Circumcenter of a Triangle

Geometry Unit 5: Lesson 3-1

Construct a ray which is the angle bisector of $\angle B$. Label the points of intersection of the first arc and the sides of $\angle B$ as $A$ and $C$. Label the point on the angle bisector where the other two arcs intersect as $D$.

What do you know about every point on the angle bisector that you constructed?

Inscribed Circle - A circle inscribed in a polygon touches each side of the polygon at exactly one point. The circle is completely inside the polygon.

Concurrent - When three or more lines intersect at the same point.

Point of concurrency - The point at which the lines intersect.

The incenter of a triangle is a point from which you can __Inscribe__ a __Circle__ inside the triangle.

If the circle touches each of the sides of the triangle once, then the point where the sides of the triangle touch are each __Equidistant__ from the incenter. They are each the length of the __Radius of the circle__ from the incenter when measured on the perpendicular.

Since we know any point that is equidistant from the sides of an angle lies on the angle bisector, we can conclude that the incenter lies on the angle bisectors of the triangle.
The incenter is located by constructing the ___________________________ of the triangle.

The angle bisectors of a triangle are ___________________ at the incenter.

A perpendicular must be constructed from the _____________ to one side of the triangle to determine the radius of the circle.

**Theorem**: The incenter is equidistant from the **perpendicular segment where the circle touches the sides of the triangle**.

Find the value of x:

1. 

2. 

Construct the line which is the perpendicular bisector of \( \overline{AB} \). Label the midpoint of \( \overline{AB} \) as \( M \). Label point \( C \) and point \( D \) on the perpendicular bisector above and below \( \overline{AB} \) respectively.

What do you know about every point on the perpendicular bisector that you constructed?
Circumscribed Circle - A circle circumscribed about a polygon passes through all the vertices of the polygon.

Concurrent - When three or more lines intersect at the same point.

Point of concurrency - The point at which the lines intersect.

The circumcenter of a triangle is a point from which you can **Circumscribe** a **Circle** around the triangle.

If all of the triangles vertices lie on the circumference of the circle, then each of the triangles vertices are **Equidistant** from the circumcenter. They are each the length of the **radius** from the circumcenter.

Since we know any point that is equidistant from the endpoints of a segment lies on the perpendicular bisector of the segment, we can conclude the circumcenter lies on the perpendicular bisectors of the sides of the triangle.

**Theorem**: The vertices of the triangle are **Equidistant** from the circumcenter.

Find the circumcenter:

1. 
2. 
1. Using a compass and straightedge, locate the incenter of $\triangle ABC$. Inscribe a circle in $\triangle ABC$.

2. Using a compass and straightedge, locate the incenter of $\triangle DEF$. Inscribe a circle in $\triangle DEF$.

3. Using a compass and straightedge, locate the incenter of $\triangle LMN$. Inscribe a circle in $\triangle LMN$. 

Name______________________
4. What do you notice about the location of the incenter in each of the three triangles (acute, right, and obtuse)?

5. The incenter is located by constructing the __________________________ of the triangle.

6. The angle bisectors of a triangle are concurrent at a point named the ________________.

7. The incenter is equidistant from the __________________________ of the triangle.

8. A perpendicular must be constructed from the ________________ to one side of the triangle to determine the radius of the circle.

1. Using a compass and straightedge, locate the circumcenter of \( \triangle ABC \). Circumscribe a circle about \( \triangle ABC \).
2. Using a compass and straightedge, locate the circumcenter of $\triangle DEF$. Circumscribe a circle about $\triangle DEF$.

3. Using a compass and straightedge, locate the circumcenter of $\triangle LMN$. Circumscribe a circle about $\triangle LMN$. 

[Diagram of Triangle DEF] 

[Diagram of Triangle LMN]
4. What do you notice about the location of the circumcenter in each of the three triangles (acute, right, and obtuse)?

5. The circumcenter is located by constructing the ____________________________ of the sides of the triangle.

6. The perpendicular bisectors of the sides of a triangle are concurrent at a point named the ________________ ________________.

7. The circumcenter is equidistant from the three ____________________________ of the triangle.
The median of a triangle is a segment whose endpoints are a vertex and the midpoint of the opposite side.

The medians are concurrent at the centroid.

The centroid is also called the center of gravity of a triangle because it is the point where a triangular shape will balance.

1. Using a compass and straightedge, locate the centroid of \( \triangle ABC \).
2. Using a compass and straightedge, locate the centroid of $\triangle DEF$.

3. Using a compass and straightedge, locate the centroid of $\triangle GHI$.

4. What do you notice about the location of the centroid in each of the three triangles (acute, right, and obtuse)?

5. Using a compass, compare the two lengths the centroid divides a median into.
6. The centroid divides a median in the ratio of ________________. The longer segment extends from a _______________ to the centroid. The shorter segment extends from the centroid to the _______________.

Examples:
1. a) In ΔXYZ, \(XA = 8\). What is the length of \(XB\)?

b) In ΔXYZ, \(ZA = 9\). What is the length of \(ZC\)?

c) What is the ratio of \(ZA\) to \(AC\)? Explain.

2. ΔABC has vertices \(A(0, 1)\), \(B(3, 6)\), and \(C(6, 1)\). What are the coordinates of the centroid of ΔABC?
3. a) For ΔPQS, is PR a median, an altitude, or neither? Explain.

b) For ΔPQS, is QT a median, an altitude, or neither? Explain.

4. For ΔABC, is each segment a median, an altitude, or neither? Explain.

a) AD

b) EG

c) CF

5. ΔABC has vertices A(1, 3), B(2, 7), and C(6, 3). What are the coordinates of the orthocenter of ΔABC?
Indirect Proof

A proof using indirect reasoning is an indirect proof.

Steps for Writing an Indirect Proof
1. State as a temporary assumption, the opposite (negation) of what you want to prove.
2. Show that this temporary assumption leads to a contradiction.
3. Conclude that the temporary assumption must be false, and that what you want to prove must be true.

1. Given: $\triangle ABC$ is scalene
   Prove: $\angle A$, $\angle B$, and $\angle C$ all have different measures

2. Given: In $\triangle RST$, $\angle R \neq \angle T$
   Prove: $RS \neq ST$
3. Given: \( \angle 1 \not\equiv \angle 3 \)
Prove: \( \angle 2 \not\equiv \angle 4 \)

4. Given: \( \overline{MO} \equiv \overline{ON}, \overline{MP} \not\equiv \overline{NP} \)
Prove: \( \angle MOP \not\equiv \angle NOP \)

Reflection:
1. Explain what you understand about an indirect proof, or what needs more clarifying.
Corollary
The measure of an exterior angle of a triangle is greater than the measure of each of its remote interior angles.

\[
\begin{array}{c|c}
\text{If . . .} & \text{Then . . .} \\
\angle 1 \text{ is an exterior angle} & m\angle 1 > m\angle 2 \text{ and} \\
& m\angle 1 > m\angle 3
\end{array}
\]

Theorems
The longest side of a triangle is opposite the largest angle.
The shortest side of a triangle is opposite the smallest angle.

Triangle Inequality Theorem
The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

\[
\begin{align*}
AB + AC &> BC \\
AB + BC &> AC \\
AC + BC &> AB
\end{align*}
\]
1. List the sides of $\triangle TUV$ in order from shortest to longest.

2. In $\triangle SOX$, $SO = 6$, $OX = 5$, $XS = 10$. List the angles in order from smallest to largest.

3. Can a triangle have sides with the given lengths? Explain.
   a) 3 ft, 7 ft, 8 ft
   b) 5 ft, 10 ft, 15 ft
   c) 2 m, 6 m, 9 m

5. Two sides of a triangular sandbox are 5 ft and 8 ft long. What is the range of possible lengths for the third side?

Reflection:
1. What do you know about the lengths of the sides of a triangle?
Inequalities in Two Triangles
Geometry Unit 5: Lesson 7

1. a) Which of the following statements must be true?
   - A) AS < YU
   - B) SK > YU
   - C) SK < YU
   - D) AK = YU

b) Write an inequality that relates LN and OQ.

2. The diagram below shows the position of a swing at two different times. As the speed of the swing ride increases, the angle between the chain and \( \overline{AB} \) increases. Is the rider farther from point A at Time 1 or Time 2? Explain how the Hinge Theorem justifies your answer.
3. a) What is the range of possible values for $x$?

b) What is the range of possible values for $x$?